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Algebraic Quantization of Causal Sets

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A scheme for an algebraic quantization of the causal sets of Sorkin *et al.* is presented. The suggested scenario is along the lines of a similar algebraization and quantum interpretation of finitary topological spaces due to Zapatrin and this author. To be able to apply the latter procedure to causal sets Sorkin's 'semantic switch' from 'partially ordered sets as finitary topological spaces' to 'partially ordered sets as locally finite causal sets' is employed. The result is the definition of 'quantum causal sets'. Such a procedure and its resulting definition are physically justified by a property of quantum causal sets that meets Finkelstein's requirement for 'quantum causality' to be an immediate, as well as an algebraically represented, relation between events for discrete locality's sake. The quantum causal sets introduced here are shown to have this property by direct use of a result from the algebraization of finitary topological spaces due to Breslav, Parfionov, and Zapatrin.

1. INTRODUCTION

An effective procedure has been developed for substituting a continuous topological space, such as a bounded region in a spacetime manifold, by a finitary one which was then seen to possess the structure of a partially ordered set (poset) (Sorkin, 1991). With every such poset an algebra, the poset's incidence algebra, was subsequently associated (Breslav *et al.*, 1999). Hence finitary substitutes for continuous topologies enjoyed a purely algebraic representation in terms of incidence algebras. Recently a quantum interpretation was given to the latter algebraized finitary topological spaces and the whole procedure was called 'algebraic quantization of discretized spacetimes' (Raptis and Zapatrin, 2000).

On the other hand, Sorkin has accounted for a significant change of physical interpretation for the aforementioned posets from ones whose partial

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order encodes topological information as in Sorkin (1991) to ones whose partial order stands for the causal, ‘after’ relation between events (Sorkin, 1989). Thus he and coworkers arrived at the notion of causal set (Bombelli *et al.* 1987; Sorkin, 1989). The new interpretation also substituted the posets’ finitarity by the causal sets’ local finiteness property. Thus it seems inevitable or at least natural that the incidence algebras associated with the topologically interpreted posets inherit this causal interpretation from the ‘semantic switch’ advocated by Sorkin (1989). The resulting algebras may be coined ‘causal incidence algebras’. Finally, if we give these algebras a quantum interpretation as in Raptis and Zapatrin (2000), we are naturally led to ‘quantum causal incidence algebras’. Then effectively we will have algebraically quantized Sorkin *et al.*’s causal sets to ‘quantum causal sets’.

These quantum causal incidence algebras are presented here as sound models of quantum causal sets. We support our claim by a result from Breslav *et al.*, 1999) that vindicates an insight of Finkelstein (1988) on how to model causality in the quantum deep.

The paper is organized as follows: in Section 2 we recall some key results from Sorkin (1991), essentially how a finitary substitute for continuous topology has the structure of a poset. In Section 3 we recall from Breslav *et al.* (1999) how to associate with every poset an algebra—the poset’s incidence algebra. In Section 4 we briefly present Sorkin’s semantic switch from ‘posets as finitary topological spaces’ to ‘posets as locally finite causal sets’ found in Sorkin (1989), and thus define causal incidence algebras. In Section 5 we select from Raptis and Zapatrin (2000) some aspects of the quantum interpretation of incidence algebras, hence lead to the structure of quantum causal incidence algebras modeling quantum causal sets. In Section 6 we use a result from Breslav *et al.* (1999) that supports the soundness of our algebraic models of quantum causal sets. The conclusion at the end sum up our approach to quantum causal sets.

2. ASPECTS OF FINITARY SUBSTITUTES

The essential result from Sorkin (1991) for our exposition here is the equivalence between finitary substitutes of bounded regions of continuous topological spaces and posets. Below we recall briefly this equivalence.

Assume a finite continuous topological space S, for instance, a bounded region of a spacetime manifold. Let S be covered by a locally finite collection $\mathcal{U}$ of bounded open sets U in the sense that each of S’s points has an open neighborhood that meets only a finite number of U’s in $\mathcal{U}$. Any two points x, y of S are indistinguishable with respect to its locally finite open cover $\mathcal{U}$ if $\forall U \in \mathcal{U}: x \in U \Leftrightarrow y \in U$. Indistinguishability with respect to this subtopology $\mathcal{U}$ of S is an equivalence relation on the latter’s points and is symbolized by $\leftrightarrow$. Taking the quotient $S/\leftrightarrow$ results in the substitution of S by equivalence classes of its points whereby two points in the same equivalence class are covered by (i.e., belong to) the same, finite in number, open neighborhoods U of $\mathcal{U}$, thus are indistinguishable by it. Call the quotient space $\mathcal{F}$.

Now let x, y be points belonging to two distinct equivalence classes in $\mathcal{F}$. Consider the smallest open sets in the subtopology $\mathcal{U}$ of S containing x and y respectively given by $\Lambda(x) := \cap\{U \in \mathcal{U}: x \in U\}$ and $\Lambda(y) := \cap\{U \in \mathcal{U}: y \in U\}$. Define the relation $\rightarrow$ between x and y as follows: $x \rightarrow y \Leftrightarrow \Lambda(x) \subset \Lambda(y) \Leftrightarrow x \in \Lambda(y)$. Then assume that $x \leftrightarrow y$ in the previous paragraph stands for $x \rightarrow y$ and $y \rightarrow x$. Now, $\rightarrow$ is a partial order on $\mathcal{F}$ and $\mathcal{S}$ has been effectively substituted by the finitary $\mathcal{F}$ which is a T_0 topological space with the structure of a poset. Sorkin uses the finitary topological and partial order-theoretic languages interchangeably exactly due to this equivalence between T_0 finitary substitutes and posets. For the future purposes of the present paper we distill this to the following statement: in Sorkin (1991) a partial order is interpreted topologically. We call it a ‘topological partial order’ and the poset encoding it a ‘topological poset’.

3. ASPECTS OF INCIDENCE ALGEBRAS

The aspect of Breslav *et al.* (1999) that is of significance here is that with every topological poset P an algebra $\Omega(P)$—the poset’s incidence algebra—is associated, so that the order-theoretic encodement of finitary substitutes has an equivalent algebraic description in terms of incidence algebras. $\Omega(P)$ as a linear space, in Dirac’s ket-bra notation,[2] is defined as

$$\Omega(P) = \mathrm{span}\{|p\rangle\langle q| : p \rightarrow q\} \tag{1}$$

with product between two of its ket-bras given by

$$|p\rangle\langle q| \cdot |r\rangle\langle s| = |p\rangle\langle q|r\rangle\langle s| = \langle q|r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & \text{if} \quad q = r \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In a so-called ‘spatialization procedure’ Breslav *et al.* brought into 1–1 corespondence the elements p of a poset P and the primitive ideals I_p of its incidence algebra $\Omega(P)$ by defining the latter as

$$I_p = \mathrm{span}\ \{|q\rangle\langle r| : |q\rangle\langle r| \neq |p\rangle\langle p|\} \tag{3}$$

and thus defined the primitive spectrum of $\Omega(P)$ as $\mathcal{S} = \{I_p\}$.

[2] The reader can also refer to the paper of Zapatrin (1998) for an early and detailed exposition of incidence algebras associated with topological posets. However, incidence algebras were not presented there in Dirac’s ket-bra notation, so in this paper we solely refer to Breslav *et al.* (1999), which first exposed them in such a way.

Then the Rota topology is defined on $\mathcal{S}$ as being generated by the following relation ρ between any two primitive ideals $I_p, I_q \in \mathcal{S}$:

$$I_p \rho I_q \Leftrightarrow I_p I_q \, (\neq I_q I_p) \overset{\neq}{\subset} I_p \cap I_q \tag{4}$$

with $I_p I_q$ their product ideal.

The central question raised and settled in Breslav *et al.* (1999) is when the Rota topology on $\mathcal{S}$ of $\Omega(P)$ is the same as the finitary one when the poset P, whose incidence algebra is $\Omega(P)$, is the finitary substitute for a continuous topological space, a topological poset, as in Section 2. The answer to this question provided an invaluable clue which is used in Section 6 for showing that a quantum causal incidence algebra has an important property, deriving from considerations of discrete locality, that any sound algebraic model of 'quantum causality' should possess. Thus we postpone its presentation until Section 6. First we need to make contact with causality by following Sorkin's paradigm of reinterpreting posets from topological to causal.

4. SORKIN'S CAUSALIZATION OF POSETS

In a change of physical interpretation, ultimately of physical theory, Sorkin stopped thinking of posets as encoding the topological information of finitary substitutes of continuous topological spaces and reinterpeted the partial order $\rightarrow$ between their elements as the causal, 'after' relation between events. In a revealing paper (Sorkin, 1989) he recalled this semantic switch of his as follows:

> . . . Still, the order inhering in the finite topological space seemed to be very different from the so-called causal order defining past and future. It had only a topological meaning but not (directly anyway) a causal one. In fact the big problem with the finite topological space was that it seemed to lack the information which would allow it to give rise to the continuum in all its aspects, not just in the topological aspect, but with its metrical (and therefore its causal) properties as well . . . The way out of the impasse involved a conceptual jump in which the formal mathematical structure remained constant, but its physical interpretation changed from a topological to a causal one . . . The essential realization then was that, although order interpreted as topology seemed to lack the metric information needed to describe gravity, the very same order reinterpreted as a causal relationship, did possess information in a quite straightforward sense . . . In fact it took me several years to give up the idea of order-as-topology and adopt the causal set alternative as the one I had been searching for . . .

This significant change of the physical semantics of the same mathematical structure, the poset, amounted to the latter being interpreted by Sorkin and coworkers as a causal set: "*a locally finite set of points endowed with a partial order corresponding to the macroscopic relation that defines past and* *future*" (Bombelli *et al.*, 1987; Sorkin, 1989). Local finiteness was defined as follows: use $\rightarrow$ of a poset P, interpreted now as a causal relation on the causal set, to redefine $\Lambda(x)$ of Section 2 for some $x \in P$ as $\Lambda(x) = \{y \in P: y \rightarrow x\}$, and dually $V(x) = \{y \in P: x \rightarrow y\}$. Now, $\Lambda(x)$ is the 'causal past' of the event x and $V(x)$ its 'causal future'. Local finiteness then requires the so-called Alexandroff set $V(x) \cap \Lambda(y)$ to be finite for all $x, y \in P$ such that $x \in \Lambda(y)$. In other words, only a finite number of events 'causally mediate' between any two events x, y, with $x \rightarrow y$, of the causal set P. Roughly, the finitarity of the topological posets of Section 2 translates by Sorkin's semantic switch to the local finiteness of causal sets, although it must be stressed that the physical theories that they support, the topological discretization of manifolds in Sorkin (1991) and causal set theory in Bombelli *et al.* (1987) and Sorkin (1989), respectively, are quite different in motivation, scope and aim (Sorkin, 1989, 1991).

Here we follow Sorkin's example and advocate a similar semantic switch from 'incidence algebras $\Omega(P)$ associated with topological posets P' to 'incidence algebras $\Omega(P)$ associated with causal sets P'. That is to say, we change physical meaning for the arrow $p \rightarrow q$ encoded in the ket-bra notation in (1) from topological to causal. Thus the proposed change of meaning is from 'topological incidence algebras' to 'causal incidence algebras'. However, in the new algebraic environment of incidence algebras, apart from the $\rightarrow$ structure of P, which is effectively encoded into the ket-bra symbol and the product of ket-bras in $\Omega(P)$ (2), a new element of structure absent from P, namely superposition $+$ of ket-bras in $\Omega(P)$, enables us to also impart quantum interpretation to $\Omega(P)$ no matter whether the latter is of topological or of causal nature. We present elements of this theory next.

5. QUANTIZATION OF INCIDENCE ALGEBRAS

In Raptis and Zapatrin (2000) a quantum interpretation to topological incidence algebras was given; thus the authors arrived at an algebraically quantized version of Sorkin's (1991) discretized spacetimes. The reader is referred to Raptis and Zapatrin (2000) for technical details. Below we only collect from it the evidence supporting this quantum interpretation of incidence algebras.

Let P be the poset corresponding to a finitary substitute of a continuous topological space in the sense of Sorkin (1991). Let $\Omega(P)$ be the incidence algebra associated with it and defined as in Section 3, Eq. (1); then:

(a) The algebraic operation $+$ between ket-bras in $\Omega(P)$ naturally enjoys a physical interpretation as coherent quantum superposition.

(b) The split of any incidence algebra $\Omega(P)$ into a commutative subalgebra $\mathscr{A}$ of grade-zero vectors $\mathscr{A} = \Omega^0 = \text{span}\{|p\rangle\langle p|: p \rightarrow p\}$ and a linear subspace $\mathcal{R} = \text{span}\,\{|p\rangle\langle q|\}_{P<q} = \Omega^1 \oplus \Omega^2 \oplus \cdots$ of vectors[3] with grade or degree ≥ 1 as $\Omega = \mathcal{A} \oplus \mathcal{R}$, naturally affords a physical interpretation as 'the algebra of quantum spacetime states'[4] and 'the space of quantum dynamical transition processes between them'[5], respectively.

(c) The quantum interpretations given in (a) and (b) above seem all the more plausible when one also interprets Sorkin's 'inverse limit' of finitary substitutes P to the continuous manifold space M that they approximate as a correspondence limit in the quantum sense of Bohr. Then $\mathcal{A}$ is expected to decohere to the classical commutative algebra of spacetime coordinates parametrizing events (classical 'position' vector states), and $\mathcal{R}$ to the classical cotangent Lie algebra of kinematical derivations of them (classical 'momentum' covector states).

What is of importance here is to borrow the quantum interpretation of topological incidence algebras from Raptis and Zapatrin (2000) and apply it to the causal incidence algebras of the previous section. Doing so, we arrive straightforwardly at the concept of 'quantum causal incidence algebras' modeling 'quantum causal sets'. The soundness of this model of quantum causal sets is shown next.

6. LOCAL ALGEBRAIC QUANTUM CAUSALITY

Finkelstein (1988) intuited that a sound quantum model of causality should essentially meet the following two conditions:

(a) Be algebraic and have a quantum interpretation for this algebraic structure.

(b) What is algebraized should not be the classical causality relation, which, like the one between the elements of Sorkin's causal sets, is usually modeled by a partial order, which in turn, being transitive (Bombelli *et al.*, 1987; Sorkin, 1989), is nonlocal (mediated). Rather, a local (immediate) version of it should be algebraically quantized. That is, the physical causal connection between events in the quantum deep should be one connecting nearest neighboring events. Symbolically, $\rightarrow$ is such that $(x \rightarrow y)$ and $\nexists z$: $x \rightarrow z \rightarrow y$.

Causal net theory was proposed as a local, discrete, algebraic, and quatum interpreted alternative to Sorkin's causal set theory that satisfies the two demands above (Finkelstein, 1988). The discrete locality aspect of causal net theory is that it can be thought of as causal set theory constrained to Alexandroff neighborhoods $V(x) \cap \Lambda(y)(x \rightarrow y)$ of zero cardinality. Its

[3] Ω^n is defined as $\Omega^n = \text{span}\,\{|p\rangle\langle q|\}_{\deg|p\rangle\langle q|=n}$, with the degree (or grade) deg of $|p\rangle\langle q|$ standing for 'the difference of cardinalities of p and q' (Raptis and Zapatrin, 2000).

[4] Called 'stationaries' in Raptis and Zapatrin (2000).

[5] Called 'transients' (Ω^1) and 'paths' (Ω^i; $i \geq 2$) in Raptis and Zapatrin (2000).

quantum algebraic aspect is that it represents causal relations between events algebraically with a quantum interpretation for this algebraic structure. In brief, causal nets are sound models of 'quantum causal spaces'. In the present paper, too, quantum causal incidence algebras are proposed as models of quantum causal sets and they plainly meet Finkelstein's requirement (a). Below we show how they also satisfy (b) in a straightforward way.

Recall the question posed at the end of Section 3, namely, when the topology encoded in a poset P that substitutes a continuous topology à la Sorkin (1991)[6] is the same as the Rota topology of its associated incidence algebra $\Omega(P)$ which is generated by the relation ρ in (4). To answer it, let us recall from Raptis and Zapatrin (2000) how a poset P may be associated with an incidence algebra Ω.

Elements of the poset $P(\Omega)$ are taken to be the irreducible representations of the algebra Ω. Building the partial order on $P(\Omega)$ consists of two steps. First, the nearest neighbor connections $p \to q$ are defined according to the following rule: let p, q be two irreducible representations of Ω; then denote by p^0, q^0 their kernels:

$$p^0 = p^{-1}(0); \qquad q^0 = q^{-1}(0)$$

which are primitive ideals in Ω. Then, the nearest neighbors $p \to q$ are defined as follows:

$$p \to q \Leftrightarrow p^0 q^0 \neq p^0 \cap q^0 \qquad (4')$$

where $p^0 q^0$ denotes the same as the product ideal $I_p I_q$ and $p^0 \cap q^0$ as the intersection ideal $I_p \cap I_q$ in (4). The resulting partial order on the set $P(\Omega)$ is obtained as the transitive closure of the nearest neighbor relation $\to$ in (4′). The topology associated with this partial order is referred to as the Rota topology.

Breslav *et al.* (1999) proved that the Sorkin topology is the same as the Rota topology exactly when $\to$ in (4′) is identified with ρ in (4), that is to say, when Rota's relation ρ is regarded as the transitive reduction of Sorkin's partial order relation $\to$. Since we have interpreted $\to$ causally, we can restate this result in a positive way in our causal context as follows: the 'immediate causal connection' ρ (between points of the primitive spectrum) is encoded more directly in the incidence algebra Ω of a finitary poset P than is its transitive closure.[7] Thus, causal incidence algebras are sound models of local causal sets and, *in extenso*, quantum causal incidence algebras of quantum causal sets, according to Finkelstein's two basic requirements presented above.

[6] We may call it 'the Sorkin topology'.
[7] P being interpreted as a causal set as in Section 4.

7. CONCLUSION

We may sum up the algebraic quantization procedure leading to quantum causal incidence algebras in the following diagram:

$$\begin{array}{ccc} \text{t-posets/incidence algebras} & \xrightarrow{q} & \text{q-t-posets/incidence algebras} \\ c\downarrow & & \downarrow c \\ \text{c-posets/incidence algebras} & \xrightarrow{q} & \text{q-c-posets/incidence algebras} \end{array}$$

picturing from the upper left corner the processes of '(c)ausalization' (causal reinterpretation à la Sorkin) of (t)opological posets (t-posets) and their topological incidence algebras to causal sets (c-posets) and their causal incidence algebras, followed by '(q)uantization' (quantum interpretation according to the Raptis–Zapatrin scheme) to quantum causal incidence algebras modeling quantum causal sets (q-c-posets). One may equivalently follow the other route and first quantize topological posets and their topological incidence algebras to quantum topological posets (q-t-posets) and their quantum topological incidence algebras and then causalize them to quantum causal sets. In this paper we took the c-followed-by-q route.

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